Introduction to Solution of Navier-Stokes Equation

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Computational Fluid Dynamics

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California State University Northridge

Homework for March 3

- Download the Excel workbook from the course web site for the sample convection problem with Pe_{cell} = 1.25
  - Shows results for Central and Upwind on separate worksheets
- Add similar worksheets to get results for Hybrid, Power Law, and QUICK
- Add error results for these algorithms to the error chart
- Any questions?

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Outline

- Review finite-volume convection
  - Central, upwind, power law, QUICK, TVD
- False diffusion
- Solving the Navier-Stokes Equations
  - Approaches
  - Grids
  - Pressure terms and the need for staggered grids
  - Derivation of momentum equations

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Review Algorithm Properties

- Conservative schemes – conserve properties in finite difference equations
  - Requires exit flux from one face to be same as input flux in adjacent cell
- Transportive schemes – have correct balance between diffusion and convection
- Accuracy – need schemes that have a good truncation error

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Review Algorithm Properties II

- Limit on coefficient magnitude for iteration schemes (boundedness)
  - Absolute value of diagonal coefficient must be greater than the sum of absolute values of all other coefficients
    - For simple equations here $|a_P| \geq |a_E| + |a_W|$
  - Deferred correction separates coefficients into two parts
    - Adjustment leaves $|a_P| \geq |a_E| + |a_W|$
    - Places part removed from adjusted coefficients into source term

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Review Convection Terms

- Steady equation with convection and diffusion terms in one dimension $\frac{d\rho u\varphi}{dx} = \frac{d}{dx}\Gamma\frac{d\varphi}{dx}$
- Steady continuity equation in one dimension $\frac{d\rho u}{dx} = 0$
- Apply finite volume approach to integrate small volume

$$\int \frac{d\rho u\varphi}{dx} dV = \int \frac{d}{dx}\Gamma\frac{d\varphi}{dx} dV \qquad \int \frac{d\rho u}{dx} dV = 0$$

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Review Integrated PDE

- Constant area result
  - Define $F = \rho u$ and $D = \Gamma/\delta x$

$$F_e\varphi_e - F_w\varphi_w = D_e(\varphi_E - \varphi_P) - D_w(\varphi_P - \varphi_W)$$

- Different approaches for ϕ_e and ϕ_w
  - Central difference, upwind, hybrid, power-law, QUICK, TVD
  - All get relations among neighbor nodes
    - Three nodes for all but QUICK

$$a_W\varphi_W - a_P\varphi_P + a_E\varphi_E = 0$$

    - Special treatment for boundary nodes

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Review Example Problem

- Constant ρ, u, and Γ with $\phi = \phi_0$ at x = 0 and $\phi = \phi_L$ at x = L

$$\frac{d\rho u\varphi}{dx} = \frac{d}{dx}\Gamma\frac{d\varphi}{dx} \Rightarrow \frac{\rho u}{\Gamma}\frac{d\varphi}{dx} = \frac{d}{dx}\frac{d\varphi}{dx}$$

- Exact solution below with plot on next slide

$$\frac{\varphi(x) - \varphi_0}{\varphi_L - \varphi_0} = \frac{e^{\frac{\rho u x}{\Gamma}} - 1}{e^{\frac{\rho u L}{\Gamma}} - 1}$$

$Pe = \rho uL/\Gamma$

$Pe_{cell} = \rho u\delta x/\Gamma = F/D$

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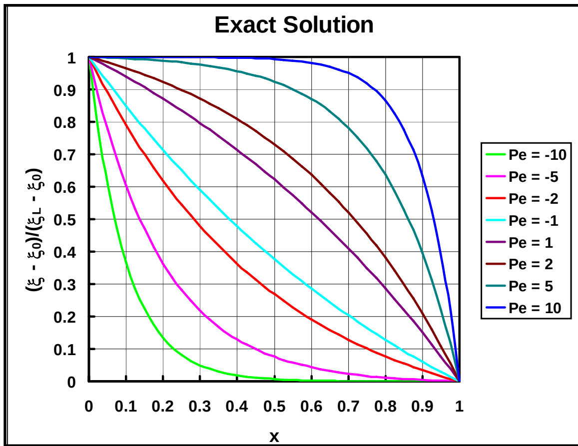

Review Central Difference

- Here δx, ρu and Γ are constants
  - $F_e = F_w = \rho u = F$ $D_e = D_w = \Gamma / \delta x = D$

$$a_W\varphi_W - a_P\varphi_P + a_E\varphi_E = \left(F + \frac{D}{2}\right)\varphi_W - 2F\varphi_P + \left(F - \frac{D}{2}\right)\varphi_E = 0$$

|-----●-----|-----●-----|-----●-----|-----●-----|-----●-----|-----●-----|

left 1 2 3 4 5 6 right

- Boundary conditions at x = 0 and x = L

$$-\left(\frac{F}{2} + 3D\right)\varphi_1 + \left(D - \frac{F}{2}\right)\varphi_2 = -(F + 2D)\varphi_{left}$$

$$\left(D + \frac{F}{2}\right)\varphi_{N-2} - \left(3D - \frac{F}{2}\right)\varphi_{N-1} = -(2D - F)\varphi_{right}$$

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Review Upwind Differences

|----------●----------|----------●----------|----------●----------|

W w P e E

- Computational formulas

$$a_W = D_w + \max(F_w, 0) \quad a_E = D_e + \max(-F_e, 0)$$
$$a_P = a_E + a_W + F_e - F_w$$

- Left boundary $a_W^* = 2D_w + \max(F_w, 0)$

$$-\left(a_E + a_W^* + F_e - F_w\right)\varphi_P + a_E\varphi_E = -a_W^*\varphi_{left}$$

- Right boundary $a_E^* = 2D_e + \max(-F_e, 0)$

$$a_W\varphi_W - \left(a_E^* + a_W + +F_e - F_w\right)\varphi_P = -a_E^*\varphi_{right}$$

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Review Hybrid Difference

- Computational Formulas

$$a_W = \max\left[F_w, \left(D_w + \frac{F_w}{2}\right), 0\right] \qquad a_E = \max\left[-F_e, \left(D_e + \frac{F_e}{2}\right), 0\right]$$
$$a_P = a_E + a_W + F_e - F_w$$

- Left boundary $a_W^* = \max[2D_w, (2D_w + F_w)]$

$$-\left(a_E + a_W^* + F_e - F_w\right)\varphi_P + a_E\varphi_E = -a_W^*\varphi_{left}$$

- Right boundary $a_E^* = \max[2D_e, (2D_e - F_e)]$

$$a_W\varphi_W - \left(a_E^* + a_W + F_e - F_w\right)\varphi_P = -a_E^*\varphi_{right}$$

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Review Power Law

- Computations $a_P = a_E + a_W + F_e - F_w$

$$a_W = D_w \max\left[0, \left(1 - |Pe_w|^5\right)\right] + \max[F_w, 0] \qquad Pe_w = F_w / D_w$$

$$a_E = D_e \max\left[0, \left(1 - |Pe_e|^5\right)\right] + \max[-F_e, 0] \qquad Pe_e = F_e / D_e$$

- Left boundary: get a^*_W with $D^*_w = 2D_w$

$$-\left(a_E + a_W^* + F_e - F_w\right)\varphi_P + a_E \varphi_E = -a_W^* \varphi_{left}$$

- Right boundary : get a^*_E with $D^*_e = 2D_e$

$$a_W \varphi_W - \left(a_E^* + a_W + F_e - F_w\right)\varphi_P = -a_E^* \varphi_{right}$$

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Review QUICK

- QUICK formulas for central node P involve five nodes instead of three

WW ww W w P e E ee EE

$$a_{WW}\varphi_{WW} + a_W \varphi_W - a_P \varphi_P + a_E \varphi_E + a_{EE}\varphi_{EE} = 0$$

$$a_W = D_w + \frac{6\alpha_w F_w + 3(1-\alpha_w)F_w + \alpha_e F_e}{8} \qquad a_{WW} = -\frac{\alpha_w F_w}{8}$$

$$a_E = D_e - \frac{3\alpha_e F_e + 6(1-\alpha_e)F_e + (1-\alpha_w)F_w}{8} \qquad a_{EE} = \frac{(1-\alpha_e)F_e}{8}$$

$\alpha_w = 1$ if $F_w > 0$ and $\alpha_e = 1$ if $F_e > 0$

$\alpha_w = 0$ if $F_w < 0$ and $\alpha_e = 0$ if $F_e < 0$

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Review QUICK II

- Boundary formulas (derivation in text)
  - Assume $F_e > 0$ and $F_w > 0$

First node from left	$a_W^* = \left(\frac{8}{3}D_w + \frac{1}{4}F_e + F_w\right) \quad a_E = \left(D_e + \frac{1}{3}D_w - \frac{3}{8}F_e\right)$ $-\left(a_E + a_W^* + F_e - F_w\right)\varphi_1 + a_E\varphi_2 = -a_W^*\varphi_{left}$
Second node from left	$a_{WW}^* = \frac{F_e}{4} \quad a_W = \left(D_w + \frac{7}{8}F_w + \frac{F_e}{8}\right) \quad a_E = D_e - \frac{3}{8}F_e$ $a_W\varphi_1 - \left(a_W + a_E + a_{WW}^* + F_e - F_w\right)\varphi_2 + a_E\varphi_3 = -a_{WW}^*\varphi_{left}$
Last node on right	$a_{WW} = \frac{F_w}{8} \quad a_W = \left(D_w + \frac{D_e}{3} + \frac{6F_w}{8}\right) \quad a_E^* = \frac{3}{8}D_e - F_e$ $a_{WW}\varphi_{N-3} + a_W\varphi_{N-2} - \left(a_{WW} + a_W + a_W^* + F_e - F_w\right)\varphi_{N-1} = -a_{WW}^*\varphi_N$

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Review TVD Algorithms

- Total Variation Diminishing schemes
  - Designed to maintain both accuracy and stability with no unphysical "wiggles"
  - Consider set of different differencing schemes for ϕ_e with positive u velocity
  - Work originated in transient gas dynamics
  - Later modifications to general CFD
  - Based on use of limiter functions that are applied to conventional formulas
  - Deferred correction required in iteration

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Review TVD Algorithms II

- General form is $\phi_e = \phi_P$ + Correction
- Correction can always be written as a correction $(\psi/2)$ times $\phi_E - \phi_P$
- Correction function ψ depends on $(\phi_P - \phi_W)/(\phi_E - \phi_P) = r$
  - $\phi_e = \phi_P + \psi(r)(\phi_E - \phi_P)/2$
- For second-order accuracy and TVD
  - For $0 < r \leq 1$, $r \leq \psi(r) \leq 2r$
  - For $1 \geq r \geq 2$, $1 \leq \psi(r) \leq r$
  - For $r > 2$, $1 \leq \psi(r) \leq 2$

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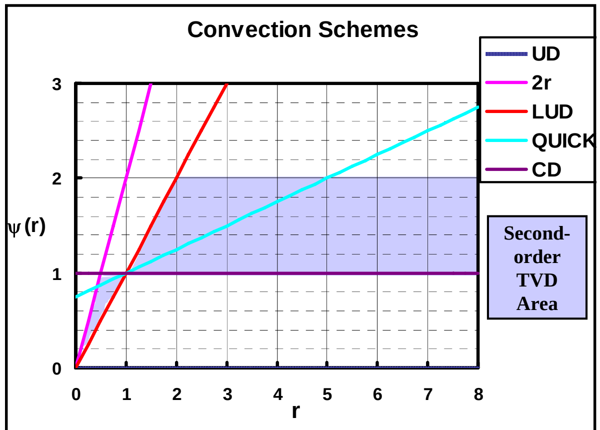

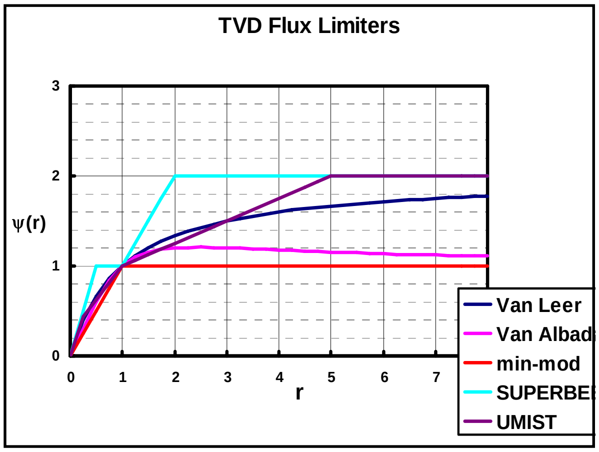

False Diffusion

- Upwind differencing causes errors similar to having a "diffusion" coefficient that is too large
- Causes smearing of results
- Especially noticeable in flows with sharp gradients and shock waves
- Effect is reduced if flow is aligned with grid (not always possible to do this)
- Different from artifical diffusion

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Navier-Stokes Equations

- Continuity and x-momentum

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho uu}{\partial x} + \frac{\partial \rho vu}{\partial y} + \frac{\partial wu}{\partial z} =$$

$$-\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\mu\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\mu\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\mu\frac{\partial u}{\partial z} + S^{(u)}$$

$$S^{(u)} = \rho B_x + \frac{\partial}{\partial x}\mu\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\mu\frac{\partial v}{\partial x} + \frac{\partial}{\partial z}\mu\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left[(\kappa - \frac{2}{3}\mu)\Delta\right]$$

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y-momentum Equation

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho uv}{\partial x} + \frac{\partial \rho vv}{\partial y} + \frac{\partial wv}{\partial z} =$$

$$-\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\mu\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\mu\frac{\partial v}{\partial y} + \frac{\partial}{\partial z}\mu\frac{\partial v}{\partial z} + S^{(v)}$$

$$S^{(v)} = \rho B_y + \frac{\partial}{\partial x}\mu\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\mu\frac{\partial v}{\partial y} +$$

$$\frac{\partial}{\partial z}\mu\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left[(\kappa - \frac{2}{3}\mu)\Delta\right]$$

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z-momentum Equation

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho uw}{\partial x} + \frac{\partial \rho vw}{\partial y} + \frac{\partial ww}{\partial z} =$$

$$-\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\mu\frac{\partial w}{\partial x} + \frac{\partial}{\partial y}\mu\frac{\partial w}{\partial y} + \frac{\partial}{\partial z}\mu\frac{\partial w}{\partial z} + S^{(w)}$$

$$S^{(w)} = \rho B_z + \frac{\partial}{\partial x}\mu\frac{\partial u}{\partial z} + \frac{\partial}{\partial y}\mu\frac{\partial v}{\partial z} +$$

$$\frac{\partial}{\partial z}\mu\frac{\partial w}{\partial z} + \frac{\partial}{\partial z}\left[(\kappa - \frac{2}{3}\mu)\Delta\right]$$

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Navier-Stokes Equations

- Continuity and momentum equations
- Up to now we have been assuming that the velocity field was known and we could find the general variable, ϕ
- This background is necessary for solving Navier-Stokes, but now we have to solve for ϕ = u, v, and w
- This gives a set of nonlinear equations (e.g., uϕ becomes uu for x-momentum)

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Navier-Stokes Equations II

- Have to find way to solve nonlinear equations
- Basic approach requires "outer" iteration process
  - Assume values for u, v, and w
  - Use these values to compute the convection/diffusion coefficients a_E, a_N, etc.
  - Solve finite difference forms of the Navier-Stokes equations for new values of u, v, and w using these "old" a_E, a_N, etc.

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Navier-Stokes Equations III

- Once new values of u, v, and w are known update a_E, a_N, *etc*. iterate again
- Consider steady-state flows now and transient flows later
- Some transient methods can use nonlinear terms at old time step to get new values for a_E, a_N, *etc*.
- For steady flows the iterations on the nonlinear terms becomes part of the overall iteration process

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Finding Pressure and Density

- For compressible flows we solve continuity and momentum for density, u, v, and w
  - Get pressure from equation of state (e.g., $p = \rho RT$) for compressible flows
- For incompressible flows find u, v, w, and p to satisfy three momentum equations and continuity
  - Density is an input parameter or depends on variables ***other than local pressure***

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Incompressible Flows

- Mach number is low (< ~0.3)
- Density (e.g., $\rho = p/RT$) may depend on mean, but not local pressure
- Can have equations like $\rho = \rho_0(1 + \beta T)$ for where $\beta = -(1/\rho)(\partial\rho/\partial T)_P$
- Density may be constant, but need not be for incompressible flows
  - Furnace flows a good example of this
- Basic idea is that density does not depend on local pressure

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Navier-Stokes Problems

- Compressible flows
  - Solve continuity and momentum for three velocity components and density
  - Get pressure from equation of state
- Incompressible flows
  - Mach number is low
  - Density is a problem input, often related to temperature (may or may not be constant)
  - Solve continuity and momentum for three velocity components and pressure

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The Steady 2D Problem

- Continuity and momentun equations
  - Have x and y direction convection-diffusion
  - Now have source term and pressure gradient

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\frac{\partial \rho uu}{\partial x} + \frac{\partial \rho vu}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\mu\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\mu\frac{\partial u}{\partial y} + S^{(u)}$$

$$\frac{\partial \rho uv}{\partial x} + \frac{\partial \rho vv}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\mu\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\mu\frac{\partial v}{\partial y} + S^{(v)}$$

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The Problem with Pressure

- Consider the x momentum equation for the u velocity component at point P

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WW ww W w P e E ee EE

- Second-order expression for pressure gradient at P $\left.\frac{\partial p}{\partial x}\right|_P = \frac{p_E - p_W}{2\delta x}$
- With this expression the velocity at P is not affected by the pressure at P
  - Also a checkerboard pattern of pressure would compute as zero pressure gradient

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The Staggered Grid

- Problem with pressure was first recognized by Harlow and Welch in 1965
- Their solution, commonly adapted for CFD until the 1990s: the staggered grid
- If the pressure p_{ij} is the pressure at (x_i, y_j) then u_{ij} is value at ($x_i - \delta x/2$, y_j) and v_{ij} is value at (x_i, $y_j - \delta y/2$)
- Colocated variables now used, but texts still introduce staggered grid

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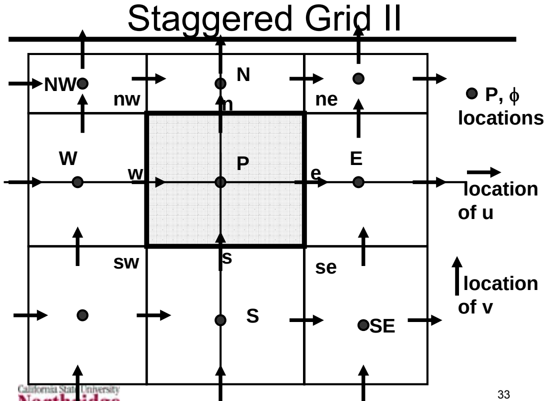
Staggered Grid II



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Staggered Grid III

- What is i,j notation for staggered grid?
- Can continue to use N, S, E, W, n, s, e, w, ne, nw, se, sw, *etc.*
- For numbering in finite-volume formulas can use i,j as p coordinate, i+1/2,j as u coordinate and i,j+1/2 as v coordinate
- Text uses I,J and i,j coordinate scheme
  - $x_i = x_I - \delta x$ and $y_j = y_J - \delta y$
  - p, other ϕ values and properties (ρ, Γ) at IJ
  - Velocity locations u_{iJ} and v_{Ij}

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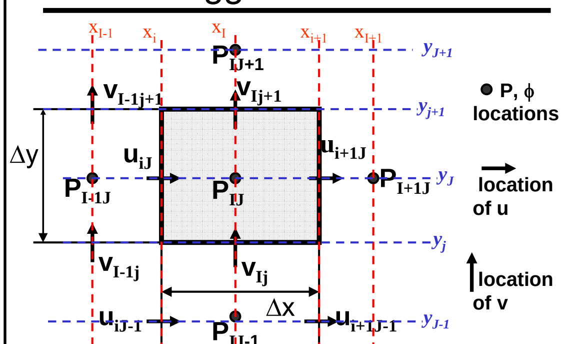
Staggered Grid IV



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Finite Volume Equations

- Extend previous results for one dimension to two dimensions
- Can use any of the difference methods discussed for convection and diffusion
  - Have four faces, n, e, s, w, in 2D
  - Apply same relations to get coefficients a_N, a_S, a_E, and a_W, using $F = \rho u$ and $D = \Gamma/\delta$ for each face
  - As before $a_P = (a_N + a_S + a_E + a_W + \Delta F)$
    - Here $\Delta F = (F_n - F_s) + (F_e - F_w)$

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Finite Volume Equations II

- Integration of pressure terms

$$\left(\int_{\Delta V}\frac{\partial p}{\partial x}dV\right)_{iJ} \approx \frac{p_{IJ}-p_{I-1J}}{x_I - x_{I-1}}(x_I - x_{I-1})A_{iJ}$$
$$= (p_{IJ}-p_{I-1J})A_{iJ} = (p_{IJ}-p_{I-1J})(y_{j+1}-y_j)\Delta z$$

$$\left(\int_{\Delta V}\frac{\partial p}{\partial y}dV\right)_{Ij} \approx \frac{p_{IJ}-p_{IJ-1}}{y_J - y_{J-1}}(y_J - y_{J-1})A_{Ij}$$
$$= (p_{IJ}-p_{IJ-1})A_{Ij} = (p_{IJ}-p_{I-1J})(x_{i+1}-x_i)\Delta z$$

37

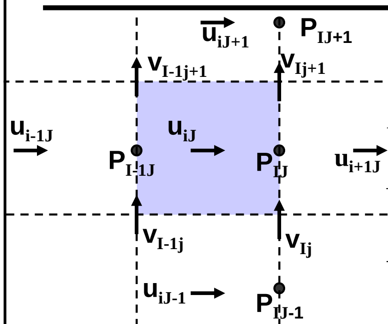
Finite Volume Equations III

- Have similar equations for u and v
- b represents integrated source term
- Note that a_K coefficients vary from node to node and are different for u and v

$$a_N u_{iJ+1} + a_S u_{iJ-1} + a_E u_{i+1J} + a_W u_{i-1J} - a_P u_{iJ}$$
$$a^{(u)}_{N_{iJ}} u_{iJ+1} \quad = (p_{iJ} - p_{i-1J})A_{iJ} + b^{(u)}_{iJ}$$
$$a_N v_{Ij+1} + a_S v_{iJ-1} + a_E v_{I+1j} + a_W v_{I-1j} - a_P v_{Ij}$$
$$a^{(v)}_{N_{Ij}} v_{Ij+1} \quad = (p_{Ij} - p_{Ij-1})A_{Ij} + b^{(v)}_{Ij}$$

38

Control Volume for u (e face)



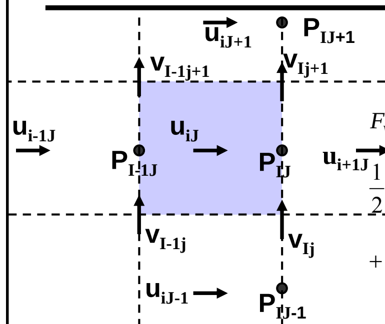
equations for e face at IJ

$$F_e = \frac{F_{I+1J}+F_{iJ}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{IJ}+\rho_{I+1J}}{2}\right)u_{i+1J} + \left(\frac{\rho_{I-1J}+\rho_{IJ}}{2}\right)u_{iJ}\right]$$

$$D_e = \frac{\Gamma_e}{x_{i+1}-x_i}$$

39

Control Volume for u (w face)



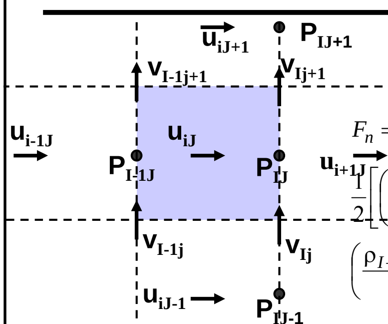
equations for w face at I-1J

$$F_w = \frac{F_{I-1J}+F_{iJ}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{I-2J}+\rho_{I-1J}}{2}\right)u_{i-1J} + \left(\frac{\rho_{I-1J}+\rho_{IJ}}{2}\right)u_{iJ}\right]$$

$$D_w = \frac{\Gamma_{I-1J}}{x_i - x_{i-1}}$$

40

Control Volume for u (n face)



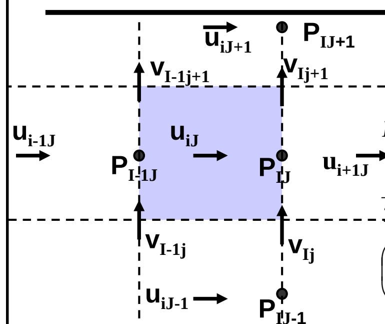
equations for n face at y_{j+1}

$$F_n = \frac{F_{Ij+1}+F_{I-1j+1}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{IJ}+\rho_{IJ+1}}{2}\right)v_{Ij+1} + \left(\frac{\rho_{I-1J}+\rho_{I-1J+1}}{2}\right)v_{I-1j+1}\right]$$

$$D_n = \frac{\Gamma_{I-1J}+\Gamma_{I-1J+1}+\Gamma_{IJ+1}+\Gamma_{IJ}}{4(y_{J+1}-y_J)}$$

41

Control Volume for u (s face)



equations for n face at y_{j+1}

$$F_s = \frac{F_{Ij}+F_{I-1j}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{IJ}+\rho_{IJ-1}}{2}\right)v_{Ij} + \left(\frac{\rho_{I-1J}+\rho_{I-1J-1}}{2}\right)v_{I-1}\right]$$

$$D_s = \frac{\Gamma_{I-1J-1}+\Gamma_{I-1J}+\Gamma_{IJ}+\Gamma_{IJ-1}}{4(y_J - y_{J-1})}$$

42

Where to Next

- Have similar equations to give various F and D terms for v control volume
- Get finite volume representation of continuity by integration over control volume centered about p_{IJ} s
- Substitute finite difference momentum equations into finite difference continuity equation to get finite difference equation for pressure
- Develop solution procedure for u, v, p